Uslovni ekstremumi

1) $u = \frac{x-y-4}{\sqrt{2}}$, uslov: $x^2+y^2=1$ $\quad (\varphi(x,y) = x^2+y^2-1$

Rj: $F(x,y) = \frac{x-y-4}{\sqrt{2}} + \lambda(x^2+y^2-1)$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 0 \\ \frac{\partial F}{\partial y} &= 0 \\ x^2+y^2 &= 1 \end{aligned}\right\} \quad \left.\begin{aligned} \frac{1}{\sqrt{2}} + 2\lambda x &= 0 \\ -\frac{1}{\sqrt{2}} + 2\lambda y &= 0 \\ x^2+y^2 &= 1 \end{aligned}\right\} \Leftrightarrow \quad \begin{aligned} &\Rightarrow x = \frac{-1}{2\lambda\sqrt{2}} \\ &y = \frac{1}{2\lambda\sqrt{2}} \\ &x^2+y^2=1 \end{aligned} \quad \begin{aligned} &\frac{1}{4\lambda^2 \cdot 2} + \frac{1}{4\lambda^2 \cdot 2} = 1 \\ &\Leftrightarrow \frac{\cancel{2}^{1}}{_4\cancel{8}\lambda^2} = 1 \\ &\lambda^2 = \frac{1}{4} \Leftrightarrow \underline{\lambda = \pm\frac{1}{2}} \end{aligned}$$

Za $\lambda_1 = \frac{1}{2} \Rightarrow x_1 = -\frac{\sqrt{2}}{2}$

$y_1 = \frac{\sqrt{2}}{2} \Rightarrow \underline{S_1\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}, \lambda_1 = \frac{1}{2}$

Za $\lambda_2 = -\frac{1}{2} \Rightarrow \underline{S_2\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)}.$

Iнаčn: $\varphi'_x = (x^2+y^2-1)' = 2x$

$\varphi'_y = 2y$

$a_{11} = \frac{\partial^2 F}{\partial x^2} = 2\lambda, \; a_{12} = 0, \; a_{22} = 2\lambda$

Za $S_1 \Rightarrow$, $\lambda_1 = \frac{1}{2}$

$$D = -\begin{vmatrix} 0 & \varphi'_x & \varphi'_y \\ \varphi'_x & a_{11} & a_{12} \\ \varphi'_y & a_{21} & a_{22} \end{vmatrix} = -\begin{vmatrix} 0 & -\sqrt{2} & \sqrt{2} \\ -\sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 1 \end{vmatrix} = 4 > 0,$$

$\Rightarrow S_1$ je uslovni min.

Za $S_2 \Rightarrow$, $\lambda = -\frac{1}{2}$

$$D = -\begin{vmatrix} 0 & \varphi'_x & \varphi'_y \\ \varphi'_x & a_{11} & a_{12} \\ \varphi'_y & a_{21} & a_{22} \end{vmatrix} = -\begin{vmatrix} 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & -1 & 0 \\ -\sqrt{2} & 0 & -1 \end{vmatrix} = -4 < 0$$

$\Rightarrow S_2$ je uslovni max.

II način:

Za $S_1 = \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right), \lambda_1 = \frac{1}{2} \Rightarrow$

$$d^2F = \frac{\partial^2 F}{\partial x^2}dx^2 + 2\frac{\partial^2 F}{\partial x \partial y}dx\,dy + \frac{\partial^2 F}{\partial y^2}dy^2 =$$

$$= 2\lambda dx^2 + 2\lambda dy^2 = 2\lambda(dx^2 + dy^2) = (dx^2 + dy^2)^2 > 0 \to$$

S_1 je ~~lokalni~~ uslovni min.

Za $S_2 = \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right), \lambda_1 = -\frac{1}{2} \Rightarrow$

$$d^2F = 2\lambda dx^2 + 2\lambda dy^2 = 2\lambda(dx^2 + dy^2) = -(dx^2 + dy^2) < 0 \to$$

S_2 uslovni maks.

II način:

$S_1, \lambda_1 \quad D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{vmatrix} = 4\lambda^2 = 1 > 0 \wedge a_{11} > 0 \to$

uslovni min.

$S_2, \lambda_2 \quad D = \begin{vmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{vmatrix} = 4\lambda^2 = 1 > 0, \; 2 \cdot \lambda = -1 < 0 \Rightarrow$

uslovni max.

2) Izračunati uslovni ekstr. fje $u = x + 2y$ uz uslov $x^2 + y^2 = 5$.

Rj. $F(x,y) = x + 2y + \lambda(x^2 + y^2 - 5)$

$$\left.\begin{array}{l} \frac{\partial F}{\partial x} = 0 \\ \frac{\partial F}{\partial y} = 0 \\ x^2 + y^2 = 5 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 1 + 2\lambda x = 0 \\ 2 + 2\lambda y = 0 \\ x^2 + y^2 = 5 \end{array}\right\} \Rightarrow \begin{array}{l} x = -\frac{1}{2\lambda} \\ y = -\frac{1}{\lambda} \end{array} \quad \to \lambda^2 = \frac{1}{4}$$

$$\frac{1}{4\lambda^2} + \frac{1}{\lambda^2} = \frac{5}{4\lambda^2} = 5 \qquad \lambda = \pm\frac{1}{2}$$

$\lambda_1 = \frac{1}{2} \Rightarrow x = -1, y = -2 \to S_1(-1, -2) \wedge \lambda_1 = \frac{1}{2}$

$\lambda_2 = -\frac{1}{2} \Rightarrow x = 1, y = 2 \to S_2(1, 2) \wedge \lambda_2 = -\frac{1}{2}$

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$$\frac{\partial^2 F}{\partial x^2} = 2\lambda, \quad \frac{\partial^2 F}{\partial x \partial y} = 0, \quad \frac{\partial^2 F}{\partial y^2} = 2\lambda$$

Za S_1 i $\lambda_1 \Rightarrow$ $D = \begin{vmatrix} 2\lambda & 0 \\ 0 & 2\lambda \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 > 0 \wedge a_{11} > 0 \Rightarrow$

S_1 - uslovni minimum.

$u_{min} = -1 - 4 = -5$

Za S_2 i $\lambda_2 \Rightarrow$ $D = \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 > 0 \wedge a_{11} < 0 \Rightarrow$ S_2 usl. maks.

$u_{max} = 1 + 4 = \underline{\underline{5}}$

Mašinstvo

Dvostruki integrali

1) Izračunajte a) $\int_1^3 \left(\int_0^2 (2x-4y)\,dy\right)dx$

b) $\int_0^1 \left(\int_0^2 (x+y)^2\,dx\right)dy$ c) $\int_1^{\ln 3} \left(\int_0^{\ln 2} e^{x-y}\,dy\right)dx$

d) $\int_0^{\ln 3} \left(\int_0^{\ln 2} e^{2x+3y}\,dx\right)dy$.

Rj.

a) $\int_1^3 \left(\int_0^2 2x\,dy - \int_0^2 4y\,dy\right)dx = \int_1^3 \left(2x\cdot y\Big|_0^2 - 4\frac{y^2}{2}\Big|_0^2\right)dx =$

$= \int_1^3 (2x\cdot 2 - 2\cdot 4)\,dx = \int_1^3 4x\,dx - 8\int_1^3 dx =$

$= 4\frac{x^2}{2}\Big|_1^3 - 8x\Big|_1^3 = 2(9-1) + 8(3-1) =$

$= 16 - 16 = 0$

b) $\int_0^1 \left(\int_0^2 (x+y)^2\,dx\right)dy = \int_0^1 \left(\int_0^2 (x^2+2xy+y^2)\,dx\right)dy =$

$= \int_0^1 \left(\frac{x^3}{3}\Big|_0^1 + 2y\frac{x^2}{2}\Big|_0^2 + y^2x\Big|_0^2\right)dy = \int_0^1 \left(\frac{8}{3} + 4y + 2y^2\right)dy =$

$= \frac{8}{3}y\Big|_0^1 + 4\frac{y^2}{2}\Big|_0^1 + 2\frac{y^3}{3}\Big|_0^1 = \frac{8}{3} + \frac{2\cdot 3}{3} + \frac{2}{3} = \frac{16}{3}$

c) d) $\int_0^{\ln 3}\left(\int_0^{\ln 2} e^{2x+3y}\,dx\right)dy = \int_0^{\ln 3}\left(\cancel{\int} e^{2x}\cdot e^{3y}\right.$

$\cancel{2x+3y=t}$ $\quad \int_0^{\ln 3}\left(e^{3y}\int_0^{\ln 2} e^{2x}\,dx\right)dy =$

$2\cdot\ln 2=\ln 4$

$= \int_0^{\ln 3}\left(e^{3y}\,\frac{1}{2}e^{2x}\Big|_0^{\ln 2}\right)dy = \frac{1}{2}\int_0^{\ln 3} e^{3y}\,(3)\,dy =$

$= \frac{3}{2}\,\frac{1}{3}\,e^{3y}\Big|_0^{\ln 3} = \frac{1}{2}(27-1) = \frac{26}{2} = 13$

(2) Zamijeniti redoslijed integracije u zadatku (1)

a) $\int_1^3\left(\int_0^2 (2x-4y)\,dy\right)dx \qquad 0\le y\le 2$

$\qquad 1\le x\le 3$

$\int_0^2\left(\int_1^3 (2x-4y)\,dx\right)dy = \int_0^2\left(2\frac{x^2}{2}-4yx\right)\Big|_1^3 dy =$

$= \int_0^2 (8-8y)\,dy = 8\,y\Big|_0^2 - 8\frac{y^2}{2}\Big|_0^2 = 16-16=0.$

c) $\int_1^{\ln 3}\left(\int_0^{\ln 2} e^{x-y}\,dy\right)dx = \int_1^{\ln 3}\left(-e^x e^{-y}\Big|_0^{\ln 2}\right)dx =$

$= \int_1^{\ln 3}\left(-e^x\left(-\frac{1}{2}\right)\right)dx = \frac{1}{2}e^x\Big|_1^{\ln 3} = \frac{1}{2}(3-e)$

$y^2 = 2x - x^2$

3) Izračunati a) $\int_{-1}^{1}\left(\int_{-x^2}^{x^2}(x^2-y)\,dy\right)dx$ ②

ⓑ $\int_{0}^{1}\left(\int_{0}^{y^2} x y^2\,dx\right)dy$ ⓒ $\int_{1}^{\ln 3}\left(\int_{0}^{x} e^{x-y}\,dy\right)dx$

ⓓ $\int_{0}^{1}\left(\int_{2y}^{2} e^{2x+3y}\,dx\right)dy$

Rj.

a) $\int_{-1}^{1}\left(x^2y-\frac{y^2}{2}\right)\Big|_{-x^2}^{x^2}dx=\int_{-1}^{1}\left(x^2\underbrace{(x^2+x^2)}_{2x^2}-\frac{1}{2}(x^4-x^4)\right)dx=$

$=2\int_{-1}^{1}x^4\,dx=2\frac{x^5}{5}\Big|_{-1}^{1}=\frac{2}{5}(1+1)=\frac{4}{5}$

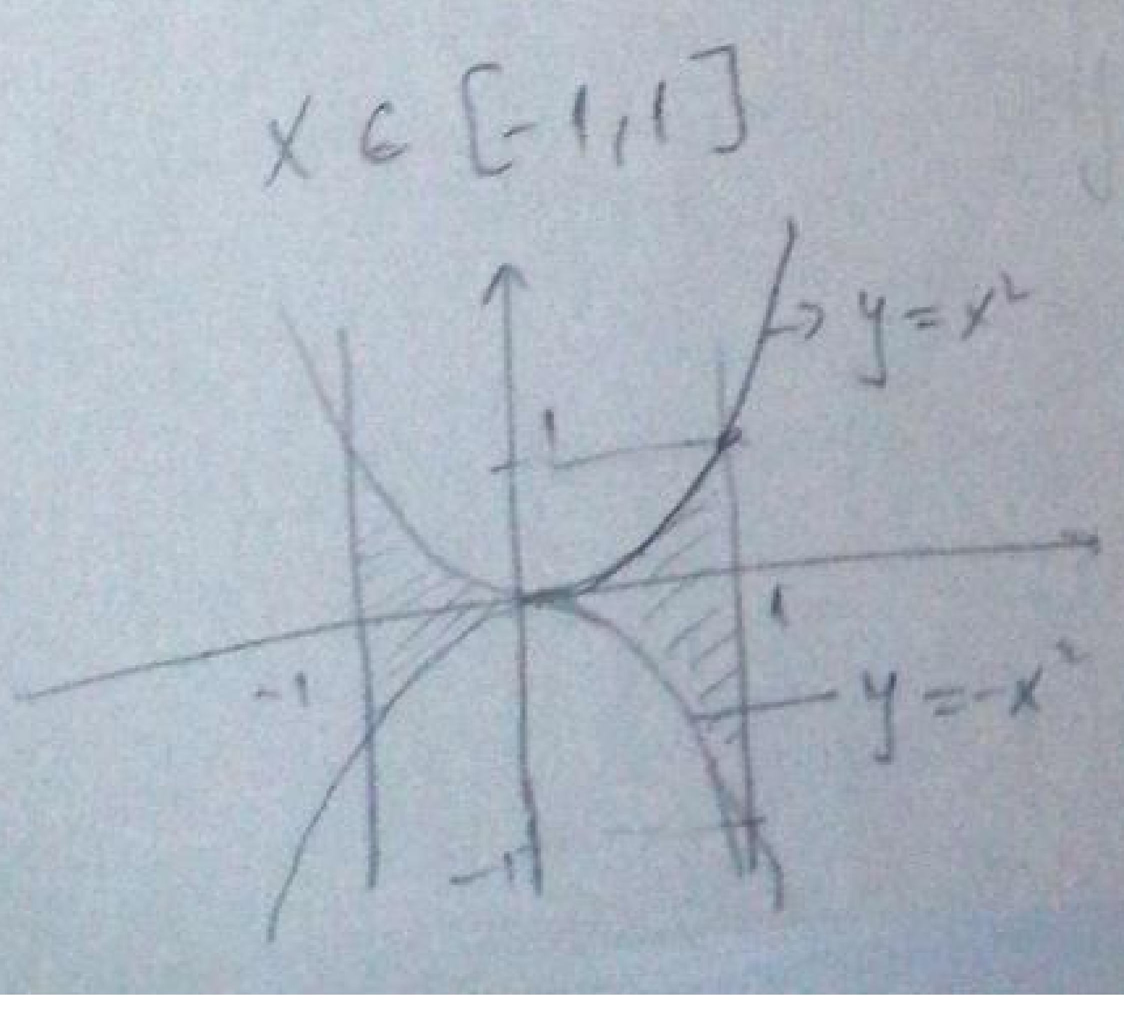
c) $\int_{1}^{\ln 3}\left(-e^x e^{-y}\Big|_0^x\right)dx=-\int_{1}^{\ln 3}e^x(e^{-x}-1)\,dx=$

$=-\int_{1}^{\ln 3}(1-e^x)\,dx=(e^x-x)\Big|_1^{\ln 3}=(3-\ln 3-e+1)=$

$=(4-\ln 3-e).$

4) Iz zadatka ① zamijeniti redosled integracije.

a) $x\in[-1,1]$ $-x^2\leq y\leq x^2$ $y\in[-1,1]$



$x^2=1$

$x=\pm 1\Rightarrow y=1$

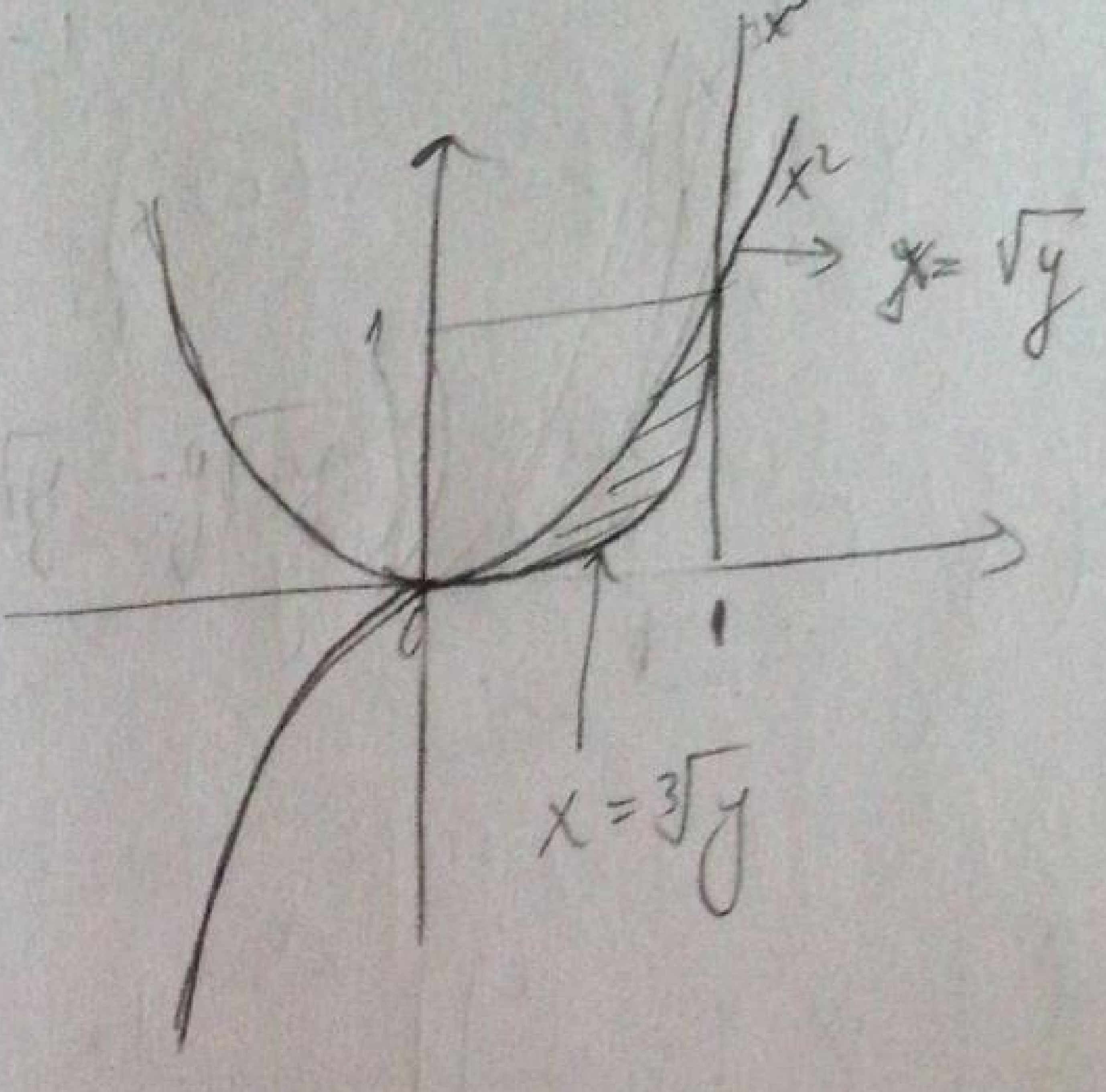
$x=\sqrt{y}$

$x^2=-y\Rightarrow x=\sqrt{-y}$

$\int_{-1}^{1}\left(\int_{\sqrt{-y}}^{\sqrt{y}}(x^2-y)\,dx\right)dy$

⑤ Zmień kolejność integracji

a) $\int_0^1 dx \int_{x^3}^{x^2} f(x,y)\,dy$



$$G = \begin{cases} 0 \le x \le 1 \\ x^3 \le y \le x^2 \end{cases}$$

$x^2 = x^3$

$x^2(1-x) = 0$

$x = 0 \lor x = 1$

$y = x^3 \Rightarrow x = \sqrt[3]{y}$

$y = x^2 \Rightarrow x = \sqrt{y}$

$$D: \left\{ (x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \quad \sqrt{y} \le x \le \sqrt[3]{y} \right\}$$

$$\int_0^1 dy \int_{\sqrt{y}}^{\sqrt[3]{y}} f(x,y)\,dy$$

b) $I = \int_0^1 dy \int_y^{3y} f(x,y)\,dx$

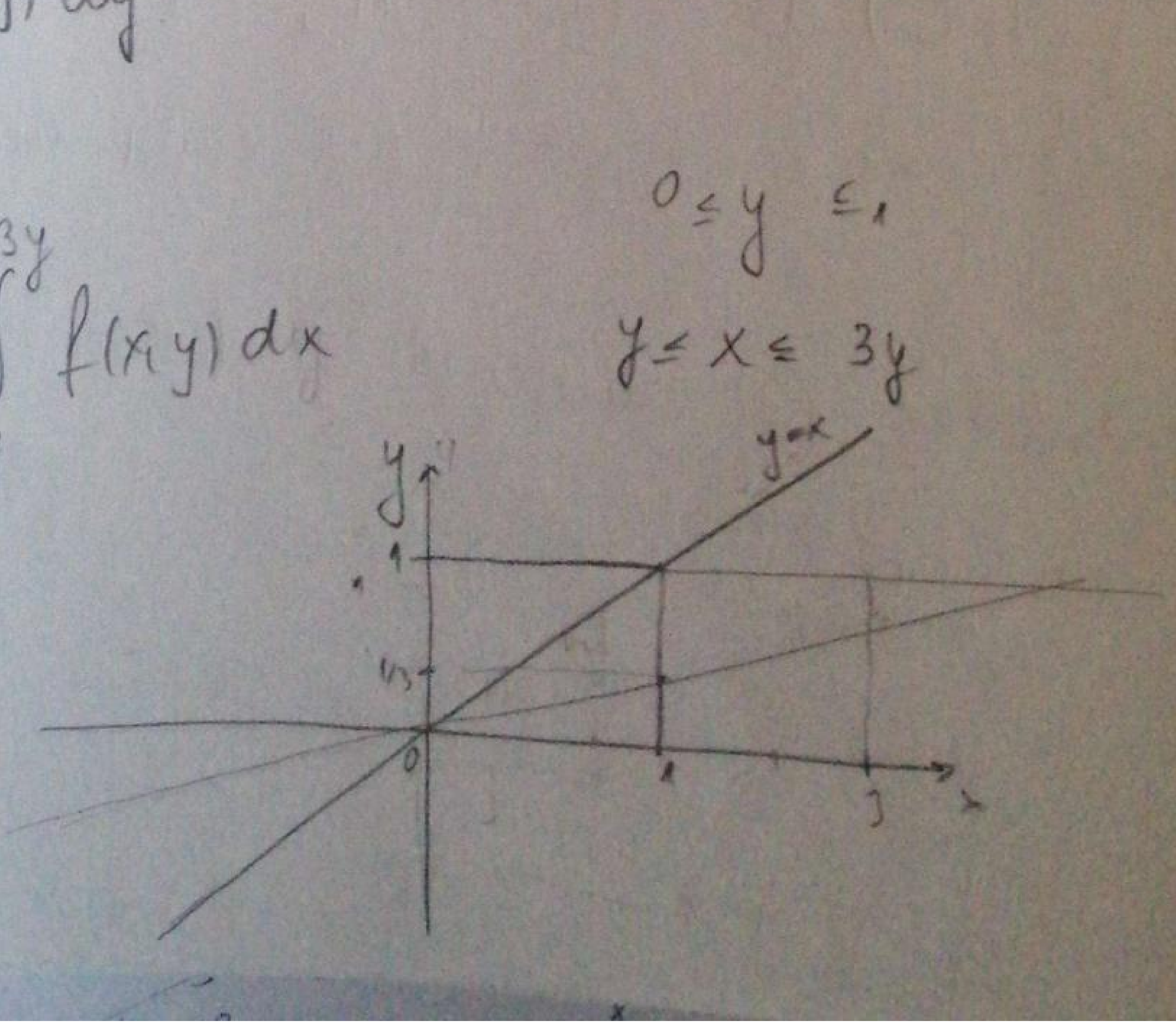
$0 \le y \le 1$

$y \le x \le 3y$

$x = y$

$x = 3y$

$y = \frac{x}{3}$



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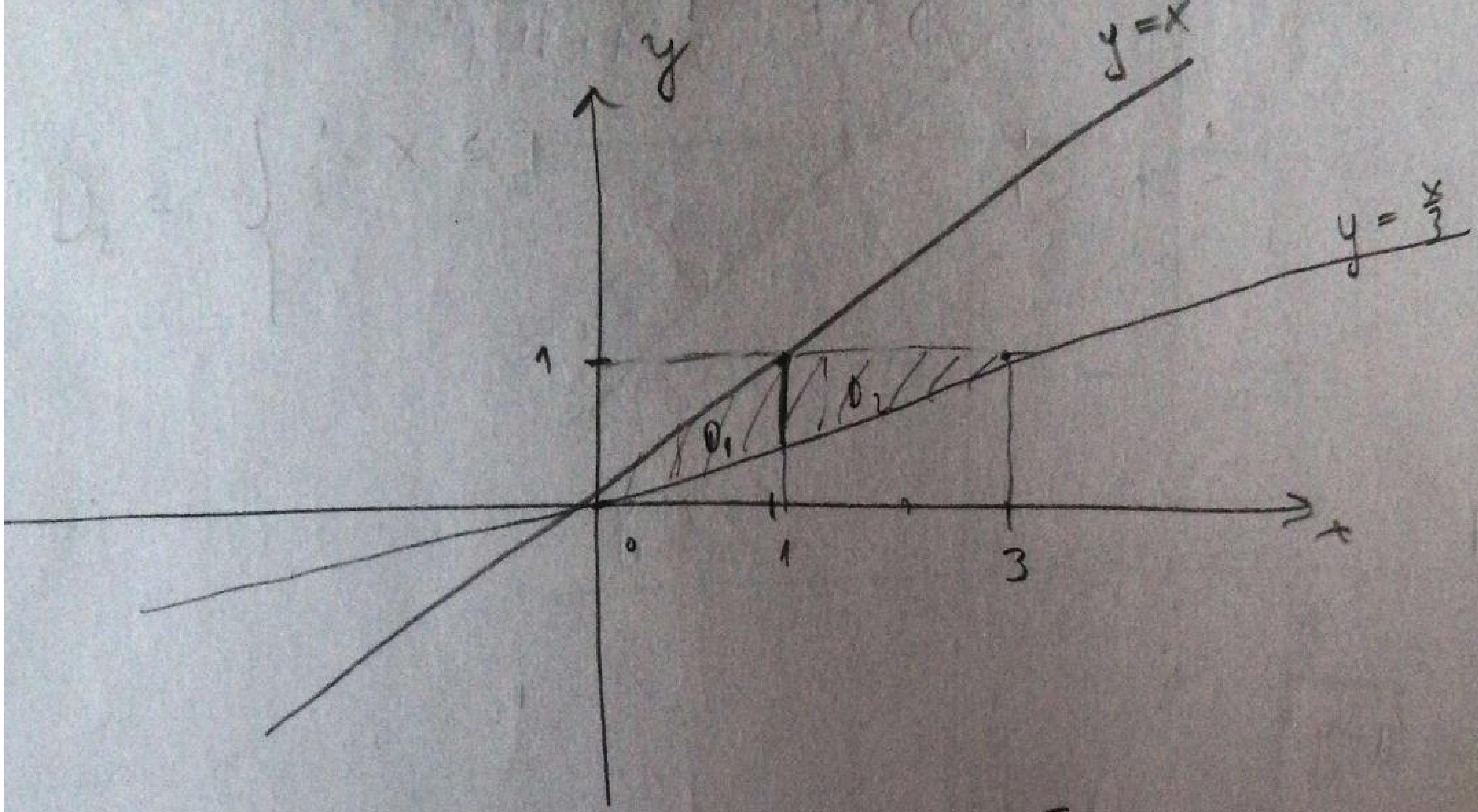
$0 \le y \le 1$ $\qquad$ $y \le x \le 3y$

$y = x$ $\qquad$ $y = \frac{x}{3}$



$D = D_1 \cup D_2$ $\qquad$ $D_1 = \begin{cases} x \in [0,1] \\ \frac{x}{3} \le y \le x \end{cases}$

$D_2 = \begin{cases} x \in [1,3] \\ \frac{x}{3} \le y \le 1 \end{cases}$

$$\int_0^1 dy \int_y^{3y} f(x,y)\,dx = \int_0^1 dx \int_{x/3}^{x} f(x,y)\,dy + \int_1^3 dx \int_{x/3}^{1} f(x,y)\,dy$$

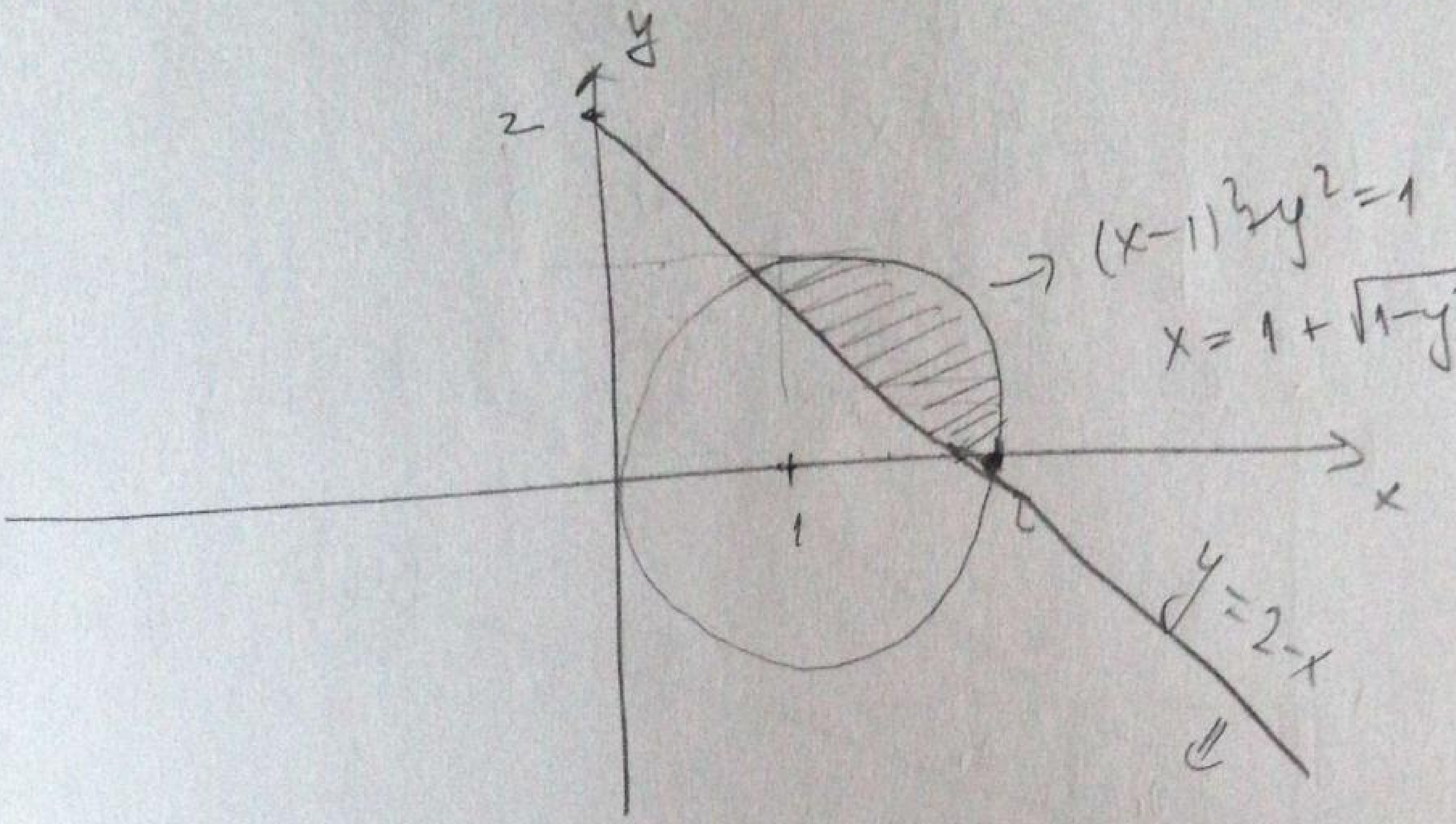
$$\int_1^2 dx \int_{2-x}^{\sqrt{2x-x^2}} f(x,y)\,dy \qquad x \in [1,2] \quad 2-x \le y \le \sqrt{2x-x^2}$$

$y = 2-x$, $\qquad y^2 = 2x - x^2$

$x^2 - 2x + y^2 = 0$

$(x-1)^2 + y^2 = 1$

$y = 2 - x$ $\qquad (x-1)^2 + y^2 = 1$ $\qquad x \in [1, 2]$



$x = 2 - y$

$0 \leq y \leq 1$

$2 - y \leq x \leq 1 + \sqrt{1-y^2}$

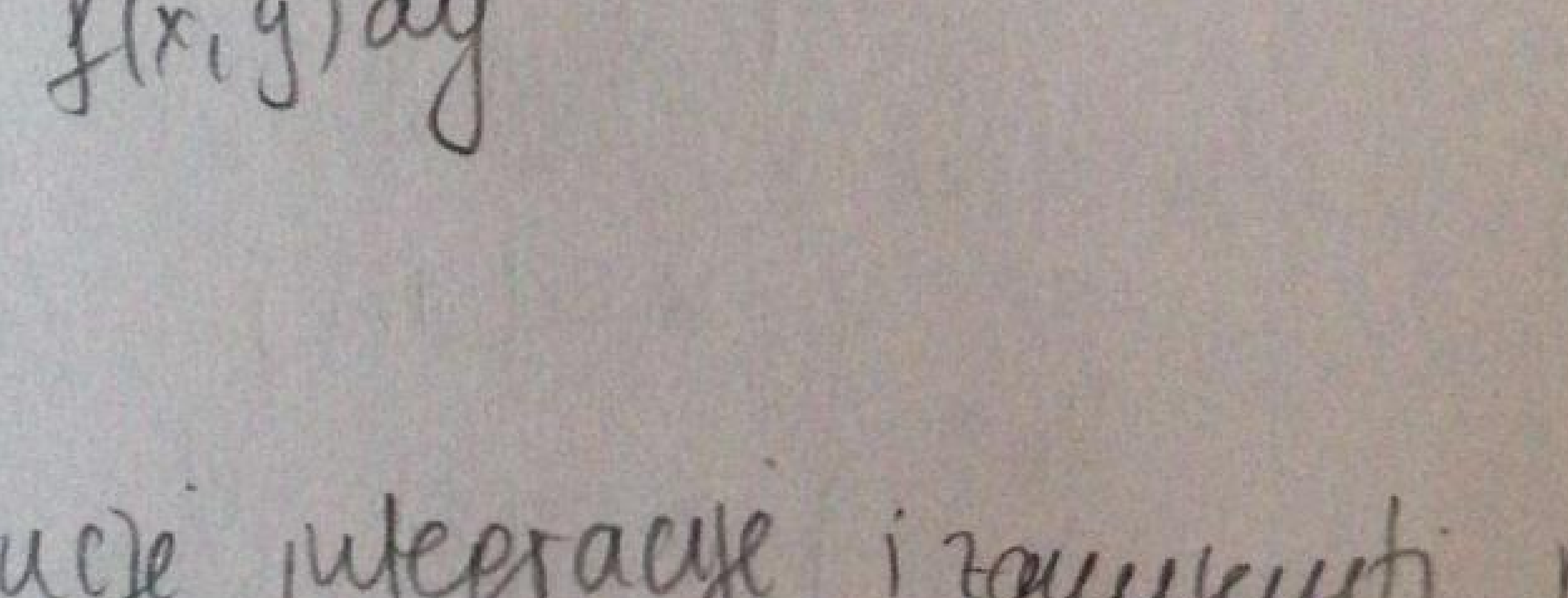
$$I = \int_0^1 dy \int_{2-y}^{1+\sqrt{1-y^2}} f(x,y)\, dy$$

4) Skicirajte područje integracije i zamijenite redoslijed integracij

a) $\int_0^2 dx \int_0^{\sqrt{x}} f(x,y)\, dy$ $\qquad$ b) $\int_0^2 dx \int_{e^x}^{e^2} f(x,y)\, dy$ $\qquad$ c) $\int_0^1 \left(\int_{y^2}^{\sqrt{y}} f(x,y)\, dx \right) dy$

Rj. a) $G = \{ (x,y) \in \mathbb{R}^2 \mid x \in [0,2],\ 0 \leq y \leq \sqrt{x} \}$

$y = 0 \qquad y = \sqrt{x}$



$y = \sqrt{x} \rightarrow x = y^2$

$2 = y^2 \Rightarrow y = \sqrt{2}$

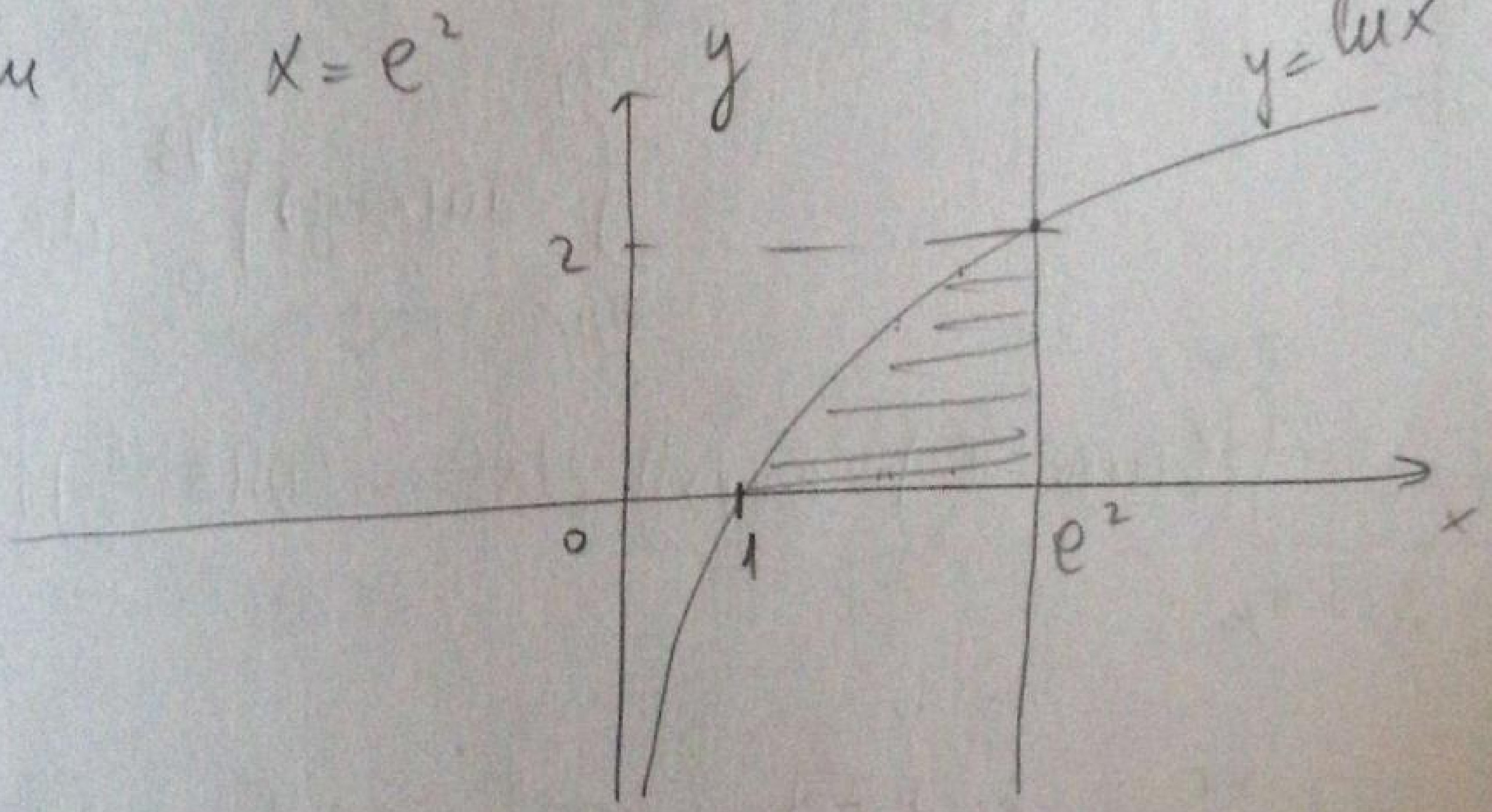
$D = \{ (x,y) \in \mathbb{R}^2 \mid 0 \leq y \leq \sqrt{2},\ y^2 \leq x \leq 2 \}$

b) $\int\limits_0^2 dy \int\limits_{e^y}^{e^2} f(x,y)\,dx$

$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; y \in [0,2] \land e^y \le x \le e^2 \right\}$$

$x = e^y \;/\ln \qquad x = e^2$

$y = \ln x$



$e^2 = e^y$

$\underline{y = 2}$

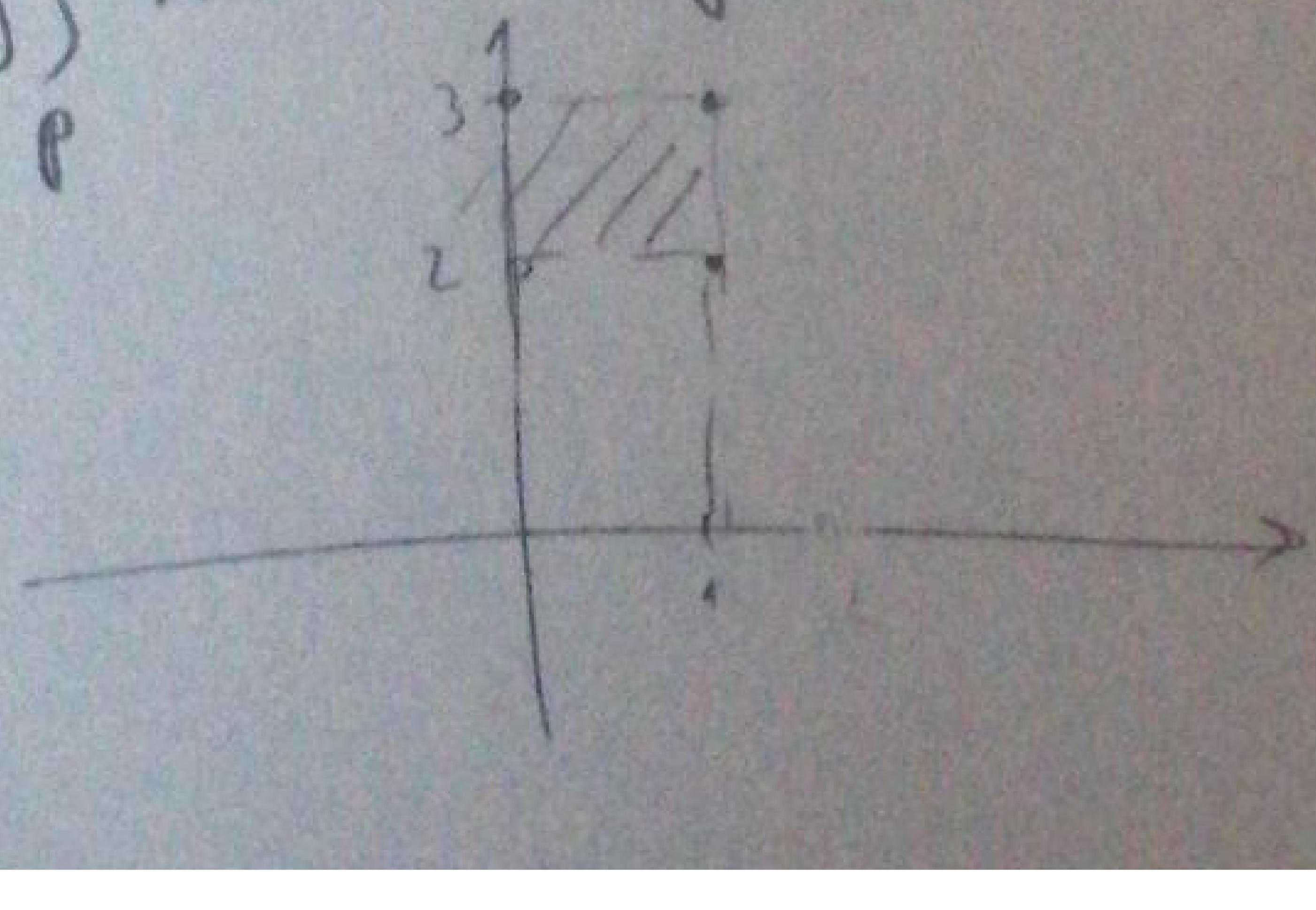
$$D = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; 1 \le x \le e^2 \land 0 \le y \le \ln x \right\}$$

$$I = \int\limits_1^{e^2} dx \int\limits_0^{\ln x} f(x,y)\,dy \ldots$$

5) Izračunati zadati integral na zadatom području:

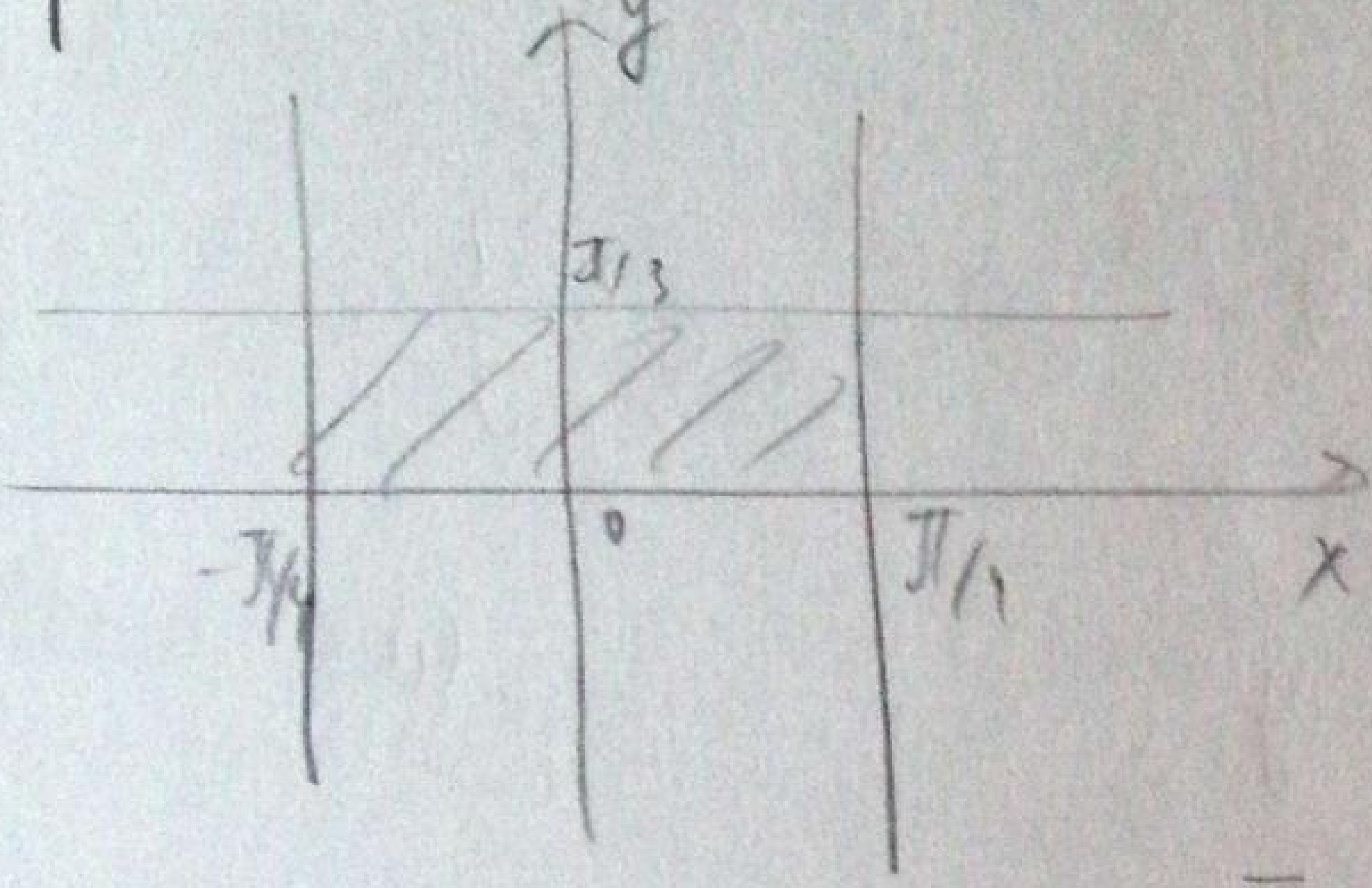
a) $\iint\limits_P x\sqrt{1-x^2}\,dx\,dy$ $\qquad$ P kvadrat sa vrhovima $(0,2)$, $(1,2)$, $(1,3)$, $(0,3)$

$\underline{P_y}$



$$\int\limits_0^1 dx \int\limits_2^3 x\sqrt{1-x^2}\,dy = \ldots$$

b) $\iint\limits_P \cos(x+y)\,dx\,dy \qquad -\frac{\pi}{4} \le x \le \frac{\pi}{2}, \quad 0 \le y \le \frac{\pi}{3}$



$$I = \int_{-\pi/4}^{\pi/4} dx \int_0^{\pi/3} \cos(x+y)\,dy =$$

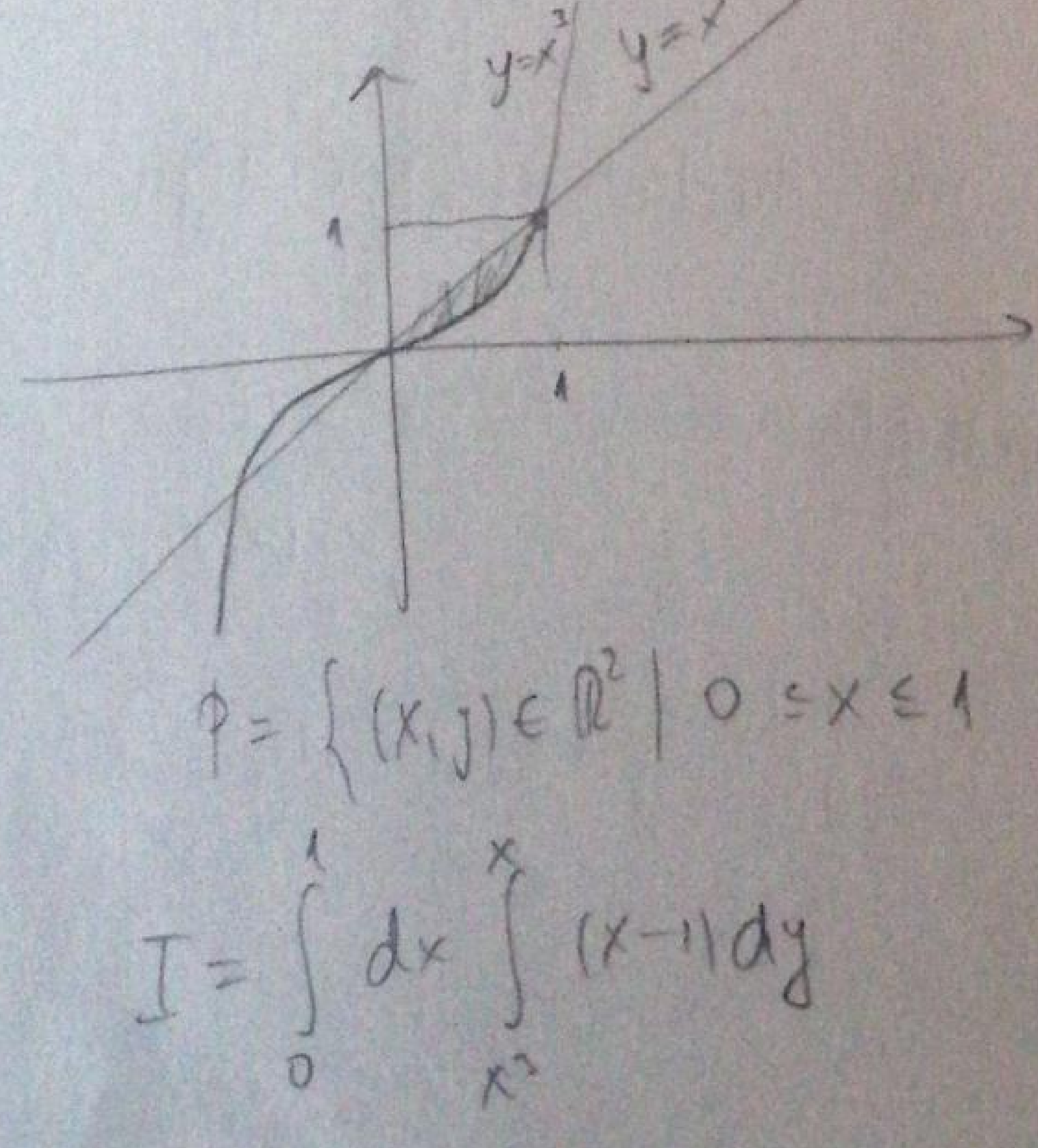
$$= \int_{-\pi/4}^{\pi/4} \sin(x+y)\Big|_0^{\pi/3} dx =$$

$$= \int_{-\pi/4}^{\pi/4} \left(\sin\left(x+\frac{\pi}{3}\right) - \sin x\right) dx = -\cos\left(x+\frac{\pi}{3}\right)\Big|_{-\pi/4}^{\pi/4} + \cos x\Big|_{-\pi/4}^{\pi/4} =$$

$$= -\cos\frac{7\pi}{12} + \cos\left(\frac{\pi}{4}\right) + \cancel{\cos\frac{\pi}{4}} - \cancel{\cos\frac{\pi}{4}} = -\cos\frac{7\pi}{12} + \frac{\sqrt{2}}{2}$$

c) $\iint\limits_P 6xy\,dx\,dy \qquad y = x, \ y = 0, \ x = \pi$

d) $\iint\limits_P (x-1)\,dx\,dy \qquad y = x \quad y = x^3 \quad y \geq 0$



$x = x^3$

$x(1-x^2) = 0$

$x = 0 \ \vee \ x = \pm 1$

$$P = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \ \wedge \ x \le y \le x^3\}$$

$$I = \int_0^1 dx \int_{x^3}^{x} (x-1)\,dy$$